1. Three squares adjoin each other as in the figure. Find the sum of angles A, B and C.

Answer 1. One plane geometry proof uses the square PQRS constructed in the figure below. From this we see that angle $A + B = 45^\circ$, so $A + B + C = 90^\circ$.

A trigonometric proof is the following:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2} + \frac{1}{2}}{1 - \frac{1}{2} \cdot \frac{1}{2}} = 1$$

so that again $A + B = 45^\circ$.

2. The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... is defined by setting the first two terms equal to 1, and thereafter by letting each term be the sum of the previous two. In other words, $a_{n+2}$ (term number $n + 2$) is defined by $a_{n+2} = a_{n+1} + a_n$ for $n = 1, 2, 3, ...$. Prove that if $n$ is divisible by $m$, then the $n$th term of the sequence is divisible by the $m$th term. For example, 8 is divisible by 4, and the 8th term (21) is divisible by the 4th term (3).

Answer 2. Fix $n$. Let us show by mathematical induction that $a_n$ evenly divides $a_{n}, a_{2n}, a_{3n} ... a_{kn}$. It is clear that $a_n$ evenly divides $a_n$. We suppose then that $a_n$ evenly divides $a_{kn}$ and try to deduce from this that $a_n$ evenly divides $a_{(k+1)n} = a_{kn+n}$. We know

$$a_{kn+2} = a_{kn+1} + a_{kn}$$

so using this rule repeatedly,

$$a_{kn+3} = 2a_{kn+1} + a_{kn} = a_3a_{kn+1} + a_2a_{kn}$$

$$a_{kn+4} = 3a_{kn+1} + 2a_{kn} = a_4a_{kn+1} + a_3a_{kn}$$

$$a_{kn+5} = 5a_{kn+1} + 3a_{kn} = a_5a_{kn+1} + a_4a_{kn}$$

and in general

$$a_{kn+p} = a_pa_{kn+1} + a_{p-1}a_{kn}.$$ 

When we reach $p = n$ we have

$$a_{kn+n} = a_na_{kn+1} + a_{n-1}a_{kn}.$$ 

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By the induction assumption, $a_n$ divides both $a_n$ and $a_{kn}$, so it divides the right hand side of this equation, thus it must divide $a_{kn+n}$.

3. Points $A$, $B$ and $C$ move counterclockwise along three coplanar circles. Each point moves with the same constant angular velocity with respect to the center of its circle. How does the centroid of triangle $ABC$ move?

Answer 3. The centers of the three circles determine a triangle whose center of gravity is a point $P$. The center of gravity of triangle $ABC$ moves with the same angular velocity around a circle with center $P$. To see this we first establish some notation. Let $C_1, C_2, C_3$ be the vectors from the origin to the centers of the circles, let $\omega$ be the constant angular velocity, let $t$ be time, let $r_1, r_2, r_3$ be the radii of the circles, and let $\phi_1, \phi_2, \phi_3$ be the initial angles of points $A, B$ and $C$. The points on the circle with center $C_n$ are obtained from

$$C_n + r_n \langle \cos(\omega t), \sin(\omega t) \rangle$$

or equivalently from

$$C_n + \cos(\omega t)(r_n i) + \sin(\omega t)(r_n j)$$

(1)

where $i$ and $j$ are the unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$, respectively. With the starting angles incorporated, the location of the center of gravity of the triangle is

$$\frac{1}{3} \sum_n (C_n + r_n \langle \cos(\omega t + \phi_n), \sin(\omega t + \phi_n) \rangle)$$

Note that $\frac{1}{3} \sum_n (C_n)$ is $P$, the vector from the origin to $P$. Now we substitute the trig identities for $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$ and factor (this is long). The result is

$$P + \cos(\omega t)n + \sin(\omega t)n^\perp$$

(2)

where $n$ is the vector

$$n = \langle r_1 \cos(\phi_1) + r_2 \cos(\phi_2) + r_3 \cos(\phi_3), r_1 \sin(\phi_1) + r_2 \sin(\phi_2) + r_3 \sin(\phi_3) \rangle$$

and $n^\perp$ is the perpendicular vector

$$n^\perp = \langle -r_1 \sin(\phi_1) - r_2 \sin(\phi_2) - r_3 \sin(\phi_3), r_1 \cos(\phi_1) + r_2 \cos(\phi_2) + r_3 \cos(\phi_3) \rangle$$

By comparing equations (2) and (1), we see the motion is around the circle with center $P$ and radius the length of $n$, and has the same angular velocity.
The solution is shorter if we introduce the complex numbers and the function \( \text{cis}(\omega t) = \cos(\omega t) + i\sin(\omega t) \). Then the \( n \)th circle can be represented as

\[
C_n + r_n \text{cis}(\omega t + \phi_n) = C_n + r_n \text{cis}(\omega t) \text{cis}(\phi_n)
\]

where now \( C_n \) is a complex number rather than a vector. The center of gravity is

\[
\frac{1}{3} \sum_{n} [C_n + r_n \text{cis}(\phi_n) \text{cis}(\omega t)]
\]

\[
= P + \frac{1}{3}[r_1 \text{cis}(\phi_1) + r_2 \text{cis}(\phi_2) + r_3 \text{cis}(\phi_3)] \text{cis}(\omega t) .
\]

Writing \( \frac{1}{3}[r_1 \text{cis}(\phi_1) + r_2 \text{cis}(\phi_2) + r_3 \text{cis}(\phi_3)] \) in the form \( r \text{cis}(\beta) \), we see that the center of gravity is located at

\[
P + r \text{cis}(\beta) \text{cis}(\omega t) = P + r \text{cis}(\omega t + \beta) .
\]

4. For a class with two or more students, show that at least two students have the same number of friends in the class. Assume that you cannot be your own friend. Also assume that if I am your friend, then you are my friend (and vice versa).

Answer 4. Suppose there are \( n \) students in the room and that everyone of them has a different number of friends. No one can have fewer than 0 friends or more than \( n - 1 \). Since there are \( n \) people in the room and exactly \( n \) numbers from 0 through \( n - 1 \), these numbers must correspond to the people in the room in some order. The person with \( n - 1 \) friends is friends with everyone in the room, including the person with 0 friends. This is a contradiction, as the person with 0 friends would then have as a friend the person with \( n - 1 \) friends.