1. The sum $S_n$ of the first $n$ terms of the sequence $a_1, a_2, \ldots$ of positive real numbers satisfies the equation $a_n + \frac{1}{a_n} = 2S_n$ for $n = 1, 2, \ldots$. Find a formula for the general term $a_n$.

Solution. Using the quadratic formula we can compute in turn $a_1 = 1$, $a_2 = \sqrt{2} - 1$, $a_3 = \sqrt{3} - \sqrt{2}$ with partial sums $S_1 = 1$, $S_2 = \sqrt{2}$, $S_3 = \sqrt{3}$. Hence we can conjecture that $a_n = \sqrt{n} - \sqrt{n-1}$ and $S_n = \sqrt{n}$. We wish to show that

$$\sqrt{n} - \sqrt{n-1} + \frac{1}{\sqrt{n} - \sqrt{n-1}} = 2\sqrt{n}, n \geq 1$$

If we rationalize the fraction,

$$\sqrt{n} - \sqrt{n-1} + \frac{\sqrt{n} + \sqrt{n-1}}{n - (n-1)} = 2\sqrt{n}$$

2. Let $P$ be a point on the bisector of an angle $\angle BAC$. Let $l$ be any line passing through $P$. Assume that $l$ intersects the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at $X$ and $Y$, respectively. Show that the quantity $\frac{1}{AX} + \frac{1}{AY}$ does not depend on the choice of $l$.

Solution by plane geometry. Solutions using trigonometry are also possible.

Construct PD parallel to AY, forming isosceles triangle $\triangle ADP$. Similarly, construct PE parallel to AX, forming the congruent isosceles triangle $\triangle AEP$. As triangles $\triangle YEP$ and $\triangle YAX$ are similar, $\frac{EP}{AY} = \frac{EY}{AY}$. Also as triangles $\triangle XDP$ and $\triangle XAY$ are similar, $\frac{DP}{AX} = \frac{DX}{AX}$. Adding,

$$\frac{EP}{AX} + \frac{DP}{AY} = \frac{EY}{AY} + \frac{DX}{AX}$$

Because triangles $\triangle ADP$ and $\triangle AEP$ are congruent and isosceles, $EP = DP = AE = AD$. So

$$EP \left( \frac{1}{AX} + \frac{1}{AY} \right) = EY \frac{AY}{AX} + DX \frac{AX}{AY} = \frac{AY - AE}{AY} + \frac{AX - AD}{AX} = 2 - AE \left( \frac{1}{AX} + \frac{1}{AY} \right)$$
Solving, we find
\[
\frac{1}{AX} + \frac{1}{AY} = \frac{2}{AE + EP} = \frac{1}{AE}
\]
which is independent of \(l\).

3. Find a formula for the sum \(1 + 11 + 111 + \cdots + \underbrace{11\cdots1}_n\).

Solution. Let \(h\) equal the sum. Expanding each term in powers of 10 we have
\[
h = 1 + (1 + 10) + (1 + 10 + 10^2) + \cdots (1 + 10 + 10^2 + \cdots + 10^{n-1})
\]
Each term in parentheses is a geometric series. Using
\[
(1 + 10 + 10^2 + \cdots + 10^k) = \frac{10^{k+1} - 1}{10 - 1} = \frac{10^{k+1} - 1}{9}
\]
we find
\[
h = \frac{10^1 - 1}{9} + \frac{10^2 - 1}{9} + \cdots + \frac{10^n - 1}{9} = \frac{(10 + 10^2 + \cdots + 10^n)}{9} - \frac{n}{9}
\]
\[
= \frac{10}{9}(1 + 10 + 10^2 + \cdots + 10^{n-1}) - \frac{n}{9} = \frac{10}{9}10^n - \frac{1}{9} - \frac{n}{9}
\]
\[
= \frac{10^{n+1} - 10 - 9n}{81}
\]

4. In a strange world there are \(n\) airports arranged around a giant circle, with exactly one airplane at each airport initially. Every day, exactly two of the airplanes fly, each going to one of its adjacent airports. Can the airplanes ever all gather at one airport?

Solution built on that of Xingping Shen, Carmel High School. Number the airports and planes \(a_1, a_2, \ldots, a_n\) and \(p_1, p_2, \ldots, p_n\) going around the circle. We will try to gather the planes at \(a_n\). This is possible if \(n\) is odd or a multiple of 4, and impossible otherwise.

Case 1: \(n\) is odd. As \(p_n\) does not need to be moved, there are an even number of planes to move. For \(1 \leq k \leq \frac{n-1}{2}\), \(p_k\) and \(p_{n-k}\) need to move the same distance. Group each \(p_k\) with \(p_{n-k}\) and each day fly one such pair one flight closer to \(a_n\).

Case 2: \(n = 2m\) and \(m\) is even. Pair all of the planes except \(p_m = p_{n/2}\) and fly them to \(a_n\) as in Case 1. Then fly \(p_m\) one step towards \(a_n\) every day, paired with one other airplane that flies alternately out of and back to \(a_n\). As \(m\) is even, they arrive at \(a_n\) on the same day.

Case 3: If \(n = 2m\) and \(m\) is odd, the solution of Case 2 doesn’t work, but we need to show there is no other solution that could work. Label airports alternately \(X\) and \(O\) around the circle. Initially the number of planes at airports marked \(X\) is \(m\), \(\#(X) = m\), and also \(\#(O) = m\). Each day, \(\#(X)\) and \(\#(O)\) either each change by 2 or remain the same, depending on which airplanes fly. So if \(m\) is odd, neither \(\#(X)\) nor \(\#(O)\) can ever become 0.