Problem 1. Your goal is to color the grey roads below, and to give a set of instructions to reach the house in the center. There should be one red and one blue road leading out of each house. There can be at most six words in the instructions, and each word must be either “red” or “blue”.

A person starting at any of the five houses must finish at the central house after completing the full set of instructions, and can visit a house more than once along the way.

Solutions to Problem 1. There are many solutions, which can be found either by trial and error, or by systematically keeping track of your choices by means of a tree diagram.

One solution is shown below. You can check that the instructions “blue”, “red”, “red”, “red”, “blue”, “blue” will lead someone from any of the houses to the house in the center.
Problem 2. Start with the number \( N_1 = 31415^{27182} \). You can produce a new number \( N_2 \) by first choosing any \( k = 1, 2, 3, 4, 5, 6, 7, 8, 9 \), then taking the product \( k \cdot N_1 \), and finally removing any 1’s appearing in the resulting decimal expansion. (If you end up with leading zeros in front of your number, remove them as well—for example 10103170 would become 370.)

Now repeat this process (choosing whatever \( k \) you like at each step) to form a sequence of numbers \( N_1, N_2, N_3, \ldots \). What is the smallest number that you can reach? How?

Solution to Problem 2. We claim that for any \( N_1 \geq 0 \) we can reach 0 by choosing some suitable sequence of \( k \)’s.

It is sufficient to show that for any integer \( N > 0 \) there is some \( k \in \{1, 2, 3, \ldots, 9\} \) so that taking the product \( k \cdot N \), and finally removing any 1’s results in a number that is strictly smaller than \( N \). (Repeating the process \( \leq N \) times, we would reach 0.)

If there are any 1’s in the decimal expansion of \( N \), then we take \( k = 1 \). After removing any 1’s from \( k \cdot N = N \), the resulting number will have fewer digits than \( N \), hence it will be strictly smaller than \( N \).

If there are no 1’s in the decimal expansion of \( N \), then there exists some \( m \geq 0 \) so that

\[
2 \cdot 10^m \leq N < 10^{m+1}.
\]

We let \( k \) be the smallest element of \( \{2, 3, \ldots, 9\} \) so that \( 10^{m+1} \leq k \cdot N \). Then, since \( k \) is the smallest element with this property, we have

\[
k \cdot N < 10^{m+1} + N.
\]

Since \( N < 10^{m+1} \), \( k \cdot N < 2 \cdot 10^{m+1} \), implying that the first digit of \( k \cdot N \) is a 1. Therefore, after removing all 1’s from the decimal expansion of \( k \cdot N \) we get a number \( N' \) satisfying

\[
N' \leq k \cdot N - 10^{m+1} < N.
\]

Problem 3. Show that there are infinitely many triples of positive integers \((a, b, n)\) with \(0 \leq a, b < n\) so that \((aa_n)^2 = bbbb_n\). Here, the repeated \( a \)'s and \( b \)'s are digits of a number, and the subscript \( n \) indicates that the number is written in base \( n \).

Solution to Problem 3. Note: for any \( n \geq 1 \), \( a = b = 0 \) is a solution. However, we consider these trivial solutions, so below we consider \( a, b > 0 \).
We’ll first find necessary conditions that the \( a, b, \) and \( n \) must satisfy, in order to narrow down the possibilities. Notice that since \( a \) and \( b \) are digits of a number written in base \( n \), we must have \( a \leq n - 1 \) and \( b \leq n - 1 \).

We have

\[
aa_n = a \cdot n + a = a(n + 1), \quad \text{so} \quad (aa_n)^2 = a^2(n + 1)^2.
\]

We also have

\[
bbbb_n = b \cdot n^3 + b \cdot n^2 + b \cdot n + 1 = b(n + 1)(n^2 + 1).
\]

So, the condition that \((aa_n)^2 = bbbb_n\) is equivalent to \(a^2(n + 1)^2 = b(n + 1)(n^2 + 1)\), which can be written more simply as

\[
a^2 = b(n^2 + 1)/(n + 1).
\] (1)

Now \( n^2 + 1 = (n + 1)(n - 1) + 2 \), so the greatest common factor between \( n^2 + 1 \) and \( n + 1 \) is either 1 or 2. If they are relatively prime, then \( b \) must be a multiple of \( n + 1 \), since the right hand side of Equation 1 is an integer. This is impossible, since \( b \leq n - 1 \). So, the greatest common factor between \( n^2 + 1 \) and \( n + 1 \) must be two and \( b \) must be a multiple of \((n + 1)/2\). Since \( 0 \leq b < n \), we conclude that \( b = (n + 1)/2 \). Therefore, \( a^2 = (n^2 + 1)/2 \), which implies

\[
2a^2 - n^2 = 1.
\] (2)

Our derivation shows that Equation 2 is a necessary conditions for \( a, b, \) and \( n \) to satisfy \((aa_n)^2 = bbbb_n\). However, you can directly check that if \( a \) and \( n \) are any solution to Equation 2 then \( a, b = (n + 1)/2, \) and \( n \) satisfy \((aa_n)^2 = bbbb_n\).

So, we must show that there are infinitely many distinct solutions to Equation 2. Some experimentation yields the following sequence of solutions for \((a, n)\):

\[
(1, 1), \quad (5, 7), \quad (29, 41), \quad (169, 239), \quad (985, 1393) \quad : 
\]

Using this sequence of solutions, you may guess the following linear recursion satisfied by the solutions

\[
(a_{i+1}, n_{i+1}) = (3a_i + 2n_i, 4a_i + 3n_i),
\] (3)
which should be used with the initial conditions \((a_1, n_1) = (1, 1)\).

We will prove by induction that \((a_i, n_i)\) is a solution to Equation (2) for each \(i \geq 1\).

For \(i = 1\), we clearly have \(2 \cdot 1^2 - 1^2 = 1\).

Suppose for some \(i\) that \((a_i, n_i)\) is a solution to Equation 2. Then,
\[
2a_{i+1}^2 - n_{i+1}^2 = 2(3a_i + 2n_i)^2 - (4a_i + 3n_i)^2
= 18a_i^2 + 24a_i n_i + 8n_i^2 - 16a_i^2 - 24a_i b_i - 9n_i^2 = 2a_i^2 - n_i^2 = 1.
\]

Therefore, by the Principle of Mathematical Induction, we conclude that for every \(i \geq 1\) that \((a_i, n_i)\) is a solution to Equation 2.

Finally, note that Equation 3 implies for each \(i\) that \(a_i, n_i \geq 1\) and therefore that \(a_{i+1} > 3a_i\), forcing the sequence of \(a_i\) to increase, thus giving us infinitely many solutions. (Similarly, the \(n_i\) are also forced to increase.)

Remark: there are many other recursive relationships possible (for example, several students discovered \(a_k = 6a_{k-1} - a_{k-2}\) and \(n_k = 6n_{k-1} - n_{k-2}\)) and the inductive proof would be similar for each.

The recursion given in Equation 3 was found by rewriting Equation 2 as
\[
(n + a\sqrt{2})(n - a\sqrt{2}) = -1.
\]

Our smallest solution, \((a, n) = (1, 1)\), corresponds to
\[
(1 + \sqrt{2})(1 - \sqrt{2}) = -1.
\]

So, for any \(i \geq 1\), we have that
\[
(1 + \sqrt{2})^{2i-1}(1 - \sqrt{2})^{2i-1} = (-1)^{2i-1} = -1.
\]

Letting
\[
n_i + a_i \sqrt{2} = (1 + \sqrt{2})^{2i-1}, \tag{4}
\]
we find our sequence of solutions \((a_i, n_i)\) given above. The recursion can then be derived from Equation 4, since
\[
n_{i+1} + a_{i+1} \sqrt{2} = (n_i + a_i \sqrt{2}) (1 + \sqrt{2})^2 = (n_i + a_i \sqrt{2}) (3 + 2\sqrt{2})
= (4a_i + 3n_i) + (3a_i + 2n_i) \sqrt{2}.
\]

**Problem 4.** Ariadne wants to design a gigantic puzzle for the 2012 Pan-Galactic Expo. She has an empty cube 2012 in. on each side, and plans to add \(k\) partitions
to form a maze. Each partition is a $1 \times 1$ in. wall or floor/ceiling separating two contiguous $1 \times 1 \times 1$ in. cells within the big cube.

The final maze is designed so that every cell is reachable from any other cell in an unique way.

Find $k$ and justify your answer.

**Solution to Problem 4.** We interpret the statement “every cell is reachable from any other cell in an unique way” to mean that for any two cells there is at most one path between them that goes through each cell at most once. Thus, our maze is allowed to have dead ends.

Let us suppose that all $6 \cdot 2012^2$ partitions on the outer faces of the big cube are already given to us. We will only count how many “interior” partitions need to be added to make the maze.

Instead of counting how many interior partitions to add, let’s suppose that all of the interior partitions are already present and count how many of them we need to remove. The total number of interior partitions is

$$3 \cdot 2011 \cdot 2012^2.$$

To see that this number is correct, notice that each interior partition is contained in some plane that intersects the cube in a total of $2012 \cdot 2012$ interior partitions. We’ll call this a “sheet” of partitions. There are 3 possible orientations of each sheet and for each orientation there are 2011 choices of sets of cells to separate.

Let’s describe the desired maze by a graph $G$ having $2012^3$ vertices, with each vertex corresponding to a cell of our maze. Two vertices will be connected by an edge if you can walk from between their corresponding cells in “one step”. (I.e. there is an edge between two vertices if the corresponding cells share a face that is not blocked by a partition.)

The condition that “every cell is reachable from any other cell in an unique way” means that $G$ is connected and has no cycles (i.e., there are no closed paths that meet every vertex other than the initial/terminal vertex exactly once). Such a graph $G$ is called a *tree*.

A proof by induction can be used to show that a tree with $n$ vertices has exactly $n - 1$ edges. (Try it yourself.)

Therefore, our tree $G$ has $2012^3 - 1$ edges. Since we are starting with all of the interior partitions, each edge corresponds to a partition that we had to remove; i.e., we must remove exactly $2013^3 - 1$ partitions.
However, we were supposed to start with an empty $2012^3$ cube and then to add partitions. The total number of “interior” partitions that one needs to add is therefore equal to:

$$3 \cdot 2011 \cdot 2012^2 - (2012^3 - 1) = 16277587025.$$  

Remark: some students expressed the answer as

$$\frac{1}{2}(8 \cdot 3 + 12 \cdot 2010 \cdot 4 + 6 \cdot 2010^2 \cdot 5 + 2010^3 \cdot 6) - (2012^3 - 1) = 16277587025.$$  

It is obtained by counting the total number of interior partitions in a different way.