Problem 1: The digits from the set \( \{1,2,3,4,5,6,7,8,9\} \) are put into a hat and pulled out sequentially and without replacement to form a 9-digit number. What is the probability that the number formed is divisible by 11? Prove that your answer is correct.

Solution for Problem 1: A number is divisible by 11 if and only if (iff) the sum the digits in odd positions minus the sum of digits in even positions is divisible by 11. If the difference is divisible by 11 (whether positive, negative, or zero), then the number is divisible by 11. For example, 1,358,027 is divisible by 11, and the difference in the sums of the odd position and even position digits is \( (1+5+0+7)-(3+8+2) = 13-13 = 0 \), which is divisible by 11. To see that this is true in general, notice that \( 100 \equiv 1 \mod{11} \), while \( 10 \equiv -1 \mod{11} \); hence, \( 10^n \equiv (-1)^n \mod{11} \), depending on whether \( n \) is odd or even. Thus,

\[
a_n a_{n-1} \cdots a_0 = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_0
\]

\[
\equiv [a_n (-1)^n + a_{n-1}(-1)^{n-1} + \cdots + a_0(-1)^0] \mod{11}.
\]

Therefore, the left-hand side is divisible by 11 iff the right-hand side is divisible by 11.

Consider the five odd position digits of the 9-digit number to be \( \{o_1,o_2,o_3,o_4,o_5\} \), the four even position digits to be \( \{e_1,e_2,e_3,e_4\} \), and denote the corresponding sums as \( S_o \) and \( S_e \) respectively. So, we want to find all possible solutions where \( S_o - S_e \) is a multiple of 11. To find the maximum difference, let the five greatest digits \( \{5,6,7,8,9\} \) fill the odd positions, and let \( \{1,2,3,4\} \) fill the even positions. Thus, the greatest difference in the sums is \( (5+6+7+8+9)-(1+2+3+4) = 25 \). Likewise, the minimum difference is \( (1+2+3+4+5)-(6+7+8+9) = -15 \). Therefore, a 9-digit number will be divisible by 11, if the difference in the sums \( (S_o - S_e) \) is either: \(-11, 0, 11, \) or \(22 \).

Note that \( S_o + S_e = 45 \), and that both sums are positive integers. Thus, we can consider:

\( S_o - S_e = -11 \) and \( S_o + S_e = 45 \) \( \Rightarrow S_o = 17 \) and \( S_e = 28 \). There are two ways to add four distinct digits between 1 and 9 to get \( S_e = 28 \): \( \{4,7,8,9\} \) and \( \{5,6,8,9\} \).

\( S_o - S_e = 0 \) and \( S_o + S_e = 45 \) does not produce integer solutions.

\( S_o - S_e = 11 \) and \( S_o + S_e = 45 \) \( \Rightarrow S_o = 28 \) and \( S_e = 17 \). There are nine ways to add four distinct digits between 1 and 9 to get \( S_e = 17 \): \( \{1,2,5,9\}, \{1,2,6,8\}, \{1,3,4,9\}, \{1,3,5,8\}, \{1,3,6,7\}, \{1,4,5,7\}, \{2,3,4,8\}, \{2,3,5,7\}, \) and \( \{2,4,5,6\} \).

\( S_o - S_e = 22 \) and \( S_o + S_e = 45 \) does not produce integer solutions.

Thus, there exists \( 2+9 = 11 \) possible sets that satisfy the conditions. The total number of sets, choosing any 4 digits from the 9, is \( C_9^4 = 126 \). Therefore, the probability is \( 11/126 \).
**Problem 2:** The parts of a regular 5-gon $ABCDE$ have areas denoted by $x$, $y$, and $z$, as shown in the figure below. If the area of $x$ is given, compute the areas $y$, $z$, and of the whole regular 5-gon.

**Solution for Problem 2:** The problem involves solving for the two unknowns $y$ and $z$, which will require a system of two equations to solve. Let $G = \overline{CE} \cap \overline{AD}$ and $H = \overline{BE} \cap \overline{AD}$, then we have

\[
\frac{|EG|}{|CG|} = \frac{\text{area of } \triangle EGD}{\text{area of } \triangle CGD} = \frac{z}{y+z}
\]

\[
= \frac{\text{area of } \triangle EGA}{\text{area of } \triangle CGA} = \frac{y+z}{x+2y}
\]

\[
= \frac{\text{area of } \triangle EGH}{\text{area of } \triangle CGH} = \frac{2y}{x+y}.
\]
It follows that

\[(y+z)^2 = (x+2y)z \]
\[z(x+y) = 2y(y+z) \]

which simplifies to

\[y^2 + z^2 = xz \quad (1)\]
\[2y^2 + yz = xz \quad (2)\]

respectively. Setting the two equations equal, yields

\[y^2 + yz - z^2 = 0.\]

Now divide through by \(z^2\), and solve the quadratic, which gives the following result

\[\left(\frac{y}{z}\right)^2 + \left(\frac{y}{z}\right) - 1 = 0\]
\[\frac{y}{z} = \frac{\sqrt{5} - 1}{2} \quad (3)\]

We have thrown out the negative solution. Now (1) and (3) yields

\[y = \frac{x}{\sqrt{5}} = \frac{\sqrt{5}x}{5}\]
\[z = \frac{(5 + \sqrt{5})x}{10}\]

We have thrown out the negative and zero solutions. Therefore, the area of the whole regular 5-gon is

\[x + 5y + 5z\]

or

\[\frac{1}{2}(7 + 3\sqrt{5})x.\]
**Problem 3:** Let $ABCD$ be the unit square, let $S$ be the circle inscribed in $ABCD$, and let $P$ be any point on the circle $S$. Consider the line segments $PA$, $PB$, $PC$, and $PD$. Let $\alpha$ be the angle between $PA$ and $PC$, and $\beta$ be the angle between $PB$ and $PD$ (see figure below). Show that $\tan^2 \alpha + \tan^2 \beta = 8$.

**Solution for Problem 3:** Let $O$ be the center of the circle and let $\phi$ be the angle between $OB$ and $OP$. First we have $BD = \sqrt{2}$, $|OB| = |OD| = \sqrt{2}/2$ and $|OP| = 1/2$, then we have

$$|BP|^2 = |OB|^2 + |OP|^2 - 2|OB||OP|\cos \phi$$

$$= \frac{3}{4} - \frac{\sqrt{2}}{2}\cos \phi.$$  

Likewise,

$$|DP|^2 = |OD|^2 + |OP|^2 - 2|OD||OP|\cos(\pi - \phi)$$

$$= \frac{3}{4} - \frac{\sqrt{2}}{2}\cos(\pi - \phi)$$

$$= \frac{3}{4} + \frac{\sqrt{2}}{2}\cos(\phi).$$

Now we have

$$\cos \beta = \frac{|BP|^2 + |DP|^2 - |BD|^2}{2|BP||DP|}$$

$$= \frac{-1}{\sqrt{9 - 8\cos^2 \phi}}.$$  

Hence, we have $\tan^2 \beta = \sec^2 \beta - 1 = 8 - 8\cos^2 \phi$. And by symmetry,

$$\tan^2 \alpha = 8 - 8\cos^2 (\phi - \pi / 2) = 8\cos^2 \phi.$$  

Hence, $\tan^2 \alpha + \tan^2 \beta = 8\cos^2 \phi + 8 - 8\cos^2 \phi = 8$. 

![Diagram](https://via.placeholder.com/150)
**Problem 4**: At a high school science competition, one of the events is the "cantilever". Each team consists of 4 students, each of whose mass is any positive real number, with the sum of the 4 student's masses adding up to at most 400 kg. The idea is to use 3 weightless boards hanging over the stage, which are each exactly 4 meters long and to arrange the students to sit on those boards. For safety's sake, only one student is allowed to be off of the stage.

The team's score is the product of the weight of the student that is not on the stage times the distance of this student from the stage. However, if the structure tips, the team scores nothing.

You may assume that each of the students is a point mass and that the students and boards may be placed at arbitrary real/fractional positions.

Determine the maximum possible score and explain the configuration of students.

For example, suppose there were only two 4-meter boards and only three students with everything arranged as follows:

Let $A$, $B$, and $C$ represent students. Note that in order for the structure not to tip, $C + \frac{1}{2}B \geq A$ otherwise the bottom plank would tip off the stage. Similarly, because $A$ and $B$ are equidistant from the end of the bottom plank, $B \geq A$.

So under the restrictions of the problem, the most weight the team could get past the stage with this configuration is 150 kg ($C = 100$ kg, $B = 150$ kg, and $A = 150$ kg). The team's final score would be $150 \text{ kg}(3\text{ m}) = 450$.

**Solution for Problem 4**: The problem assumes that for each board, there is exactly one student sitting on it (with the exception of the top one). Moreover, we assume that the student not on the stage is to the right of the stage. Also, it is obvious that given the weights of the students, the optimal solution will always have the three students on stage at the left edge of each board. Finally, we must have the student off-stage sitting on the right edge of the top board, which we will take without proof.

Let $0 < l_1 \leq l_2 \leq l_3 < 4$ denote the distances from the right ends of the bottom, middle, and top boards to the right edge of the stage, respectively. Finally, denote the weights of the
students as $A$, $B$, $C$, and $D$ where $A$ is the weight of the student sitting on the bottom board, and so on. From the conditions, we have the following:

\begin{align*}
A + B + C + D &= 400 \quad (1)
\end{align*}

\begin{align*}
(4 - l_3 + l_2)C &\geq (l_3 - l_2)D \quad (2)
\end{align*}

\begin{align*}
(4 - l_2 + l_3)B + (4 - l_3 + l_1)C &\geq (l_3 - l_1)D \quad (3)
\end{align*}

\begin{align*}
(4 - l_1)A + (4 - l_2)B + (4 - l_3)C &\geq (l_3 - l_2)D \quad (4)
\end{align*}

The goal is to maximize $l_3D$: This will occur only if all inequalities are in fact equalities (we state this without proof, but it is not too difficult). Hence, we have:

\begin{align*}
A + B + C + D &= 400 \quad (5)
\end{align*}

\begin{align*}
(4 - l_3 + l_2)C &= (l_3 - l_2)D \quad (6)
\end{align*}

\begin{align*}
(4 - l_2 + l_3)B + (4 - l_3 + l_1)C &= (l_3 - l_1)D \quad (7)
\end{align*}

\begin{align*}
(4 - l_1)A + (4 - l_2)B + (4 - l_3)C &= l_3D \quad (8)
\end{align*}

From Equation 6,

\begin{align*}
(l_3 - l_1)D - (4 - l_3 + l_1)C &= (l_2 - l_1)D + (l_2 - l_1)C
\end{align*}

\begin{align*}
&= D(l_2 - l_1)\left(1 + \frac{l_3 - l_2}{4 - l_3 + l_2}\right)
\end{align*}

\begin{align*}
&= \frac{4(l_2 - l_1)D}{4 - l_3 + l_2}
\end{align*}

whence

\begin{align*}
B &= \frac{4(l_2 - l_1)D}{(4 - l_3 + l_2)(4 - l_2 + l_1)} \quad (9)
\end{align*}

by Equation 7. From Equations 6, 7, and 9,

\begin{align*}
l_3D - (4 - l_3)C - (4 - l_2)B &= l_1(B + C + D)
\end{align*}

\begin{align*}
&= l_1D \cdot \frac{4(l_2 - l_1) + 4(4 - l_2 + l_1)}{(4 - l_3 + l_2)(4 - l_2 + l_1)}
\end{align*}

\begin{align*}
&= \frac{16l_1D}{(4 - l_3 + l_2)(4 - l_2 + l_1)}
\end{align*}

where

\begin{align*}
A &= \frac{16l_1D}{(4 - l_3 + l_2)(4 - l_2 + l_1)(4 - l_1)} \quad (10)
\end{align*}
by equation 8. Finally 5, 6, 9, and 10 yield

\[ A + B + C + D = 400 \quad \iff \quad (11) \]

\[ D \left( \frac{16l_1 + 4(l_3 - l_l)(4-l_l) + 4(4-l_2 + l_l)(4-l_l)}{(4-l_3 + l_2)(4-l_2 + l_l)(4-l_l)} \right) = 400 \quad \iff \quad (12) \]

\[ \frac{25}{4} l_3 (4-l_3 + l_2)(4-l_2 + l_l)(4-l_l) = l_3 D, \quad (13) \]

so the problem becomes maximizing the right-hand side of Equation 13. To do this, we first hold \( l_3 \) and \( l_2 \) constant. Considering only those terms with \( l_1 \) in them, we have

\[ (4-l_3 + l_2)(4-l_l) = 16 - 4l_2 + l_2 l_1 - l_1^2 = -\left( l_1 - \frac{l_2}{2} \right)^2 + 16 - 4l_2 + \frac{l_2^2}{4} \leq \left( 4 - \frac{l_2}{2} \right)^2, \]

with equality when \( 2l_1 = l_2 \). Now, holding \( l_3 \) constant and using \( l_1 = l_2 / 2 \), we consider only those terms with \( l_2 \) in them. Our goal is then to maximize

\[ (4-l_3 + l_2)\left( 4 - \frac{l_2}{2} \right)^2. \]

We can do this using calculus, but we will instead use the AM-GM inequality. Let \( u = 4 - l_3 + l_2 > 0 \) and \( v = 4 - l_2 / 2 > 0 \). Then \( u + 2v = 12 - l_3 \), from which

\[ 4 - \frac{l_3}{3} = \frac{u + v + v}{3} \geq u^{1/3}, v^{2/3} \iff \left( 4 - \frac{l_3}{3} \right) \geq uv^2, \]

with equality when \( u = v \), i.e., \( 3l_2 = 2l_3 \). Finally, we have

\[ l_3 \left( 4 - \frac{l_3}{3} \right)^3 \leq \left( \frac{l_3}{4} + 3 \frac{4 - l_3 / 3}{4} \right)^4 = 81 \]

by a similar application of AM-GM. Hence, the maximum value of \( l_3 D \) is

\[ \frac{25}{4} (8) = \frac{2025}{4}. \]

The configuration for which this is achieved is as follows:

\[ l_1 = 1, \ l_2 = 2, \ \text{and} \ l_3 = 3 \ \text{with} \]

\[ A = 100, \ B = 75, \ C = \frac{225}{4}, \ \text{and} \ D = \frac{675}{4}. \]

**Remark:** It is instructive to consider the problem with one board and two people as well as the problem with two boards and three people. With induction, one can solve this problem in the general case with \( n \) masses and \( n-1 \) boards.