**Qualifying exam for numerical analysis (Spring 2019)**

Show your work for full credit. If you cannot solve some part, attempt the subsequent parts.

1. Consider approximating $f \in C^{mn+1}$ using a linear combination of degree $mn$ polynomial basis functions $b(x)$ and $a_{i,j}(x)$:

$$\tilde{f}(x) = b(x) \int_{x_0}^{x_1} f(x)dx + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{i,j}(x) f^{(i)}(x_j)$$

Discuss reasonable conditions to use for finding $b(x)$ and $a_{i,j}(x)$, use those conditions to find formula for $b(x)$ and $a_{m-1,j}(x)$, and prove that the error for such an $\tilde{f}$ is given by:

$$\text{error}(x) = \tilde{f}(x) - f(x) = -\frac{f^{(mn+1)}(\xi)}{(mn+1)!} p(x)$$

where $p(x)$ is an appropriately defined polynomial.

2. Consider solving the following system of equations:

$$x + \alpha y^3 = 11/18$$
$$\beta x^3 + y = 7/12.$$

$x_c$ and $y_c$ are solutions to these equations for a given $\alpha$ and $\beta$. Do not assume that any parameters are large or small.

(a) Find a formula for the condition number when $x_c$ and $y_c$ are functions of $\alpha$ and $\beta$. Clearly identify which norms you use here. Give a condition that when satisfied corresponds to an ill-conditioned problem.

(b) Consider using the following iterative method to find $x_c$ and $y_c$ for a given $\alpha$ and $\beta$.

$$x_k = 11/18 - \alpha y_{k-1}^3$$
$$y_k = 7/12 - \beta x_k^3$$

Find the condition number associated with finding $x_k$ and $y_k$ given $x_{k-1}$ and $y_{k-1}$.

(c) Notice the above iterative method can be written as:

$$x_k = 11/18 - \alpha(7/12 - \beta x_{k-1}^3)^3$$

Find a formula for the condition number associated with finding $x_k$ given $x_0$. Give conditions that will result in initial machine storage errors on $x_0$ going to zero as $k \to \infty$.

3. Given $f(-h), f(h), f'(-h)$, and $f'(h)$ for $f(x) \in C^2[-h, h]$:

(a) Find a quadratic polynomial, $\tilde{f}$, that minimizes $||\tilde{f} - f||$ where $||g||^2 = \langle g, g \rangle$ and $\langle f, g \rangle = f(-h)g(-h) + f(h)g(h) + f'(-h)g'(-h) + f'(h)g'(h)$.

(b) Using part a) show that:

$$f'(0) \approx \frac{f'(-h)h}{2h^2+2} - \frac{f(-h)}{2h^2+2} + \frac{f(h)}{2h^2+2} + \frac{f'(h)h}{2h^2+2}$$

(c) Find a formula for the error of the form:

$$\text{error} = \frac{f'(-h)h}{2h^2+2} - \frac{f(-h)}{2h^2+2} + \frac{f(h)}{2h^2+2} + \frac{f'(h)h}{2h^2+2} - f'(0) = g(h)f^{(j)}(\xi)h^k$$

where you have clearly defined $g(h)$. What are the values of $j$ and $k$?

4. Assume $f(x) \in C^2[0,1]$ and consider a quadrature rule of the form

$$\int_0^1 f(x)x^{-1/3}dx \approx A_0 f(0) + A_1 f(1).$$

(a) Find $A_0, A_1$ that maximize the degree of polynomial for which the method is exact.

(b) For the values of $A_0$ and $A_1$ found in part (a), find a function $K(t)$ such that:

$$\text{error}(f) = A_0 f(0) + A_1 f(1) - \int_0^1 f(x)x^{-1/3}dx = \int_0^1 f''(t)K(t)dt$$

(c) Using appropriate justification and part (b) show that

$$\text{error}(f) = \frac{9}{80}f''(\xi), \xi \in (0,1)$$

5. Suppose $f(x)$ has a repeated root $\alpha$ with multiplicity $m$. Consider finding this root using:

$$x_k = x_{k-1} - c\frac{f(x_{k-1})}{f'(x_{k-1})}$$

(a) Find a condition on $c$ so that the above method is convergent.

(b) Show the method converges quadratically for a particular value of $c$ and give that value.

6. $\mathbf{A} \in \mathbb{C}^{m\times m}$ is unitarily diagonalizable with positive eigenvalues $\lambda_i$ and unit length eigenvectors $\hat{x}_i$. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

(a) Consider using the following iterative method to solve $\mathbf{A}\vec{x} = \vec{b}$:

$$\mu\mathbf{I}\vec{x}^{(k)} = (\mu\mathbf{I} - \mathbf{A})\vec{x}^{(k-1)} + \vec{b}$$

Prove the method converges when $\mu$ is chosen correctly. Give a condition on $\mu$ that guarantees convergence. Give a condition on $\mu$ ($\mu = f(\lambda_1, \lambda_m)$) that will approximately maximize convergence speed.

(b) Consider the iterative method:

$$\left(a\mathbf{I} - b\sum_{i=1}^{\ell}\hat{x}_i\hat{x}_i^*\right)\vec{x}^{(k)} = \left(\left(a\mathbf{I} - b\sum_{i=1}^{\ell}\hat{x}_i\hat{x}_i^*\right) - \mathbf{A}\right)\vec{x}^{(k-1)} + \vec{b}$$

Give two equations which you can solve for $a$ and $b$ that will approximately maximize convergence speed. DO NOT SOLVE THE EQUATIONS.

## Qualifying exam for numerical analysis (Spring 2017)

Show your work for full credit. If you are unable to solve some part, attempt the subsequent parts.

1. Let $f(x) = ax - 2^{-x}$ with $a$ near 1.
   (a) Find a polynomial interpolant of $f$ that passes through interpolation points at $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.
   (b) Use the interpolant to find an approximate solution to $f(x) = 0$ for $x \in [-1, 1]$.
   (c) Find the relative condition number of $g(a) \to x_0$ where $x_0$ is the approximate solution to $f(x) = 0$.

2. Consider estimating the integral $\int_0^2 e^{-x/2}dx$ using Romberg integration based on the trapezoidal rule. Increasing numbers of intervals are used (1, 2, 4, 8,...) until the desired error tolerance is met. Give the number of intervals at which we can guarantee that the absolute error associated with the best Romberg integration estimate is less than 1/100. Justify your answer by finding appropriate error bounds. Use the Euler-Maclaurin formula which holds for $h = b - a$ and any integer $m \geq 1$:

$$\int_a^b f(x)dx = h\frac{(f(b) + f(a))}{2} + \sum_{k=1}^{m-1} A_{2k}h^{2k}\left(f^{(2k-1)}(a) - f^{(2k-1)}(b)\right) - A_{2m}h^{2m+1}f^{(2m)}(\xi)$$

$$A_2 = 1/12; A_4 = -1/720; A_6 = 1/30240; \ldots; \xi \in [a, b]$$

3. $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices with complex entries ($m \geq n$). $\mathbf{C}$ and $\mathbf{\Lambda}$ are $n \times n$ matrices with complex entries. $\mathbf{A} = \mathbf{B\Lambda C}^*$, $\mathbf{BB}^* = \mathbf{I}$, $\mathbf{CC}^* = \mathbf{I}$, and $\mathbf{\Lambda}$ is diagonal with $\lambda_{jj} = \alpha_j + \beta_j i, j = 1..n, i = \sqrt{-1}$. Use the vector and matrix-induced 2-norm below when a vector and/or matrix norm is needed.
   (a) Consider the reduced singular value decomposition of $\mathbf{A} = \mathbf{U\Sigma V}^*$. Find $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{\Sigma}$ in terms of the matrices, columns, and/or components of $\mathbf{B}$, $\mathbf{\Lambda}$, and $\mathbf{C}$.
   (b) Give the matrix condition number of $\mathbf{A}$.
   (c) Show the relative condition number of $\vec{f}(\vec{x}) = \mathbf{A}\vec{x}$ at $\vec{x} = \sum_{j=1}^{n} \gamma_j \vec{c}_j = \mathbf{C}\vec{\gamma}$ is given by:

$$\kappa = \frac{\sqrt{\max_j\left(\alpha_j^2 + \beta_j^2\right)}\sqrt{\sum_j \gamma_j^2}}{\sqrt{\sum_j^n \left(\alpha_j^2 + \beta_j^2\right)\gamma_j^2}}$$

   Here $\vec{c}_j$ are the columns of the matrix $\mathbf{C}$.
   (d) For what inputs $\vec{x}$ do you expect input perturbations to result in the i) largest resulting relative errors after matrix multiplication? ii) smallest resulting relative errors after matrix multiplication?

4. Suppose that $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and that $\vec{y} \in \mathbb{R}^n$ satisfies $\vec{F}(\vec{y}) = \vec{y}$. Further suppose that all of the eigenvalues,$\lambda_i$, of its Jacobian matrix satisfy $|\lambda_i^2 - 2\lambda_i - 3| < 1$ with the Jacobian matrix being Hermitian ($\mathbf{A} = \mathbf{A}^*$). Show that the iteration defined by $\vec{x}_{n+1} = -(\vec{F}(\vec{F}(\vec{x}_n)) - 2\vec{F}(\vec{x}_n) - 3\vec{x}_n)/4$ will converge to $\vec{y}$ if $\vec{x}_0$ is close enough to $\vec{y}$. Estimate the rate of convergence.
5. Consider a real symmetric matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \ldots, \lambda_m$ and let $\mu$ be a real number such that $|\lambda_1 - \mu| = |\lambda_2 - \mu| > ... > |\lambda_{m-1} - \mu| = |\lambda_m - \mu| > 0$. Let $\vec{x}_k = (\mathbf{A} - \mu\mathbf{I})^{-1}\vec{x}_{k-1}$.
   (a) Find the value to which $\nu_k$ converges when

$$\nu_k = \frac{\vec{x}_{k+2}^T \vec{x}_k}{\vec{x}_k^T \vec{x}_k}$$

   Clearly state conditions on $\vec{x}_0$ to ensure this convergence.

   (b) What is the rate of convergence of $\nu_k$?
6. Consider approximating the solution to the differential equation $y'(x) = \lambda y(x)$ with the constraint that $y(0) = 1$.
   (a) Find a matrix equation that can be solved in order to find the quadratic polynomial approximant $p(x)$ that results from minimizing the residual

$$||p'(x) - \lambda p(x)||_2$$

   where the 2-norm is defined on the interval $[-h, h]$ and $p(x)$ is chosen so that $p(0) = 1$. The components of the matrix and vector may include unevaluated integrals. It should, however, be clear from your solution what integrands need to be integrated and how the solutions of your equation would fit into $p(x)$.

   (b) Assume that the quadratic polynomial approximant from part (a) is given by:

$$p(x) = 1 + \frac{6(10 - h^2\lambda^2)}{3h^4\lambda^4 - 16h^2\lambda^2 + 60}\lambda x + \frac{5(6 - h^2\lambda^2)}{3h^4\lambda^4 - 16h^2\lambda^2 + 60}\lambda^2 x^2$$

   Show that the error associated with this polynomial approximant at a location $x$ can be written as

$$\text{error}(x) = a_1\lambda x + a_2(\lambda x)^2 + a_3(\lambda x)^3 + O((\lambda x)^4)$$

   by comparing the approximant with the exact solution to $y'(x) = \lambda y(x)$. Include expressions for $a_1, a_2, a_3$ in terms of $h$ and $\lambda$.

   (c) Use your solution from part (b) to discuss all things that can be done to decrease the error associated with this approximation.

**Qualifying exam for numerical analysis (Spring 2017)**

Show your work for full credit. If you are unable to solve some part, attempt the subsequent parts.

1. Consider the following finite difference:

$$f'(0) \approx A_0 f(0) + A_1 f(h) + A_2 f(2h).$$

   (a) Find the weights $A_0$, $A_1$, and $A_2$ that make this method as accurate as possible.

   (b) Find the maximum degree polynomial for which this method is exact.

2. $\mathbf{B} = \mathbf{A}^*\mathbf{A} \in \mathbb{R}^{m\times m}$ has the eigenvalues $\lambda_1 > \cdots > \lambda_m$ and corresponding eigenvectors $\vec{x}_1, ..., \vec{x}_m$. Here $^*$ denotes the complex conjugate transpose of a matrix. Find the reduced singular value decomposition of $\mathbf{A}$ in terms of $\lambda_1, ..., \lambda_m, \vec{x}_1, ..., \vec{x}_m$ and $\vec{w}_1, ..., \vec{w}_m$ where $\vec{w}_1 = \mathbf{A}\vec{x}_1, ..., \vec{w}_m = \mathbf{A}\vec{x}_m$.

3. For your condition number calculations below you may choose any consistent set of matrix/vector/scalar norms but you must clearly indicate which norms you choose.

   (a) Find the relative condition number for $\vec{f}(x,y) = \begin{bmatrix} -\sin(xy) \\ y^3 \end{bmatrix}$ at $x = \pi$ and $y = 1$.

   (b) Find the matrix condition number for $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & \gamma \end{bmatrix}$. Based on your answer, when do you expect to encounter difficulties when trying to solve $\mathbf{A}\vec{x} = \vec{b}$?

4. A composite quadrature rule has error associated with it in the following form

$$\text{error} = \int_a^b f(x)dx - \sum_{ij} A_i(h) f(x_i(h) + jh) = a_1 h + a_3 h^3 + \cdots .$$

   $i$ indexes the nodal locations from the original quadrature rule, $j$ indexes each of the subintervals used in the composite rule, and $h$ corresponds to the size of each subinterval.

   (a) Obtain an estimate for the first and second Richardson extrapolants when the subinterval size is repeatedly halved. Do not use any memorized formulas, derive your results by hand.

   (b) Use your results to fill in the table below where $R(i,j)$ is the $j^{\text{th}}$ Richardson extrapolant and $i$ corresponds to the level of subinterval refinement used.

| h | $R(i,0)$ | $R(i,1)$ | $R(i,2)$ |
|---|---|---|---|
| 1 | 2 | Nothing in this cell. | Nothing in this cell. |
| 1/2 | 3 | | Nothing in this cell. |
| 1/4 | 4 | | |

5. $\tilde{f}(x)$ is the least squares approximation to $f(x)$ using a set of linearly independent basis functions $\{\phi_i(x)\}$. It is given by $\tilde{f}(x) = \sum_i c_i\phi_i(x)$ where the coefficients $c_i$ are found by solving the corresponding set of normal equations. Justify the steps in your proofs below.

   (a) Prove that $\left\langle f - \tilde{f}, \phi_j \right\rangle = 0$, $\forall j$ where $\langle f, g\rangle$ is an inner product associated with a 2-norm for functions.

   (b) Prove that $\tilde{f}$ minimizes $||f - \tilde{f}||_2$ by showing that $\forall g(x) = \sum_i d_i\phi_i(x)$ with $d_j \neq c_j$ for at least some $j$, $||f - \tilde{f}||_2 < ||f - g||_2$. Hint: It may help to rewrite $g$ as $= (\tilde{f}+$ nice terms).

6. Consider an interval $J = \{x : |x - \alpha| \leq \rho\}$, an $x_0 \in J$, and the corresponding sequence defined by $x_{i+1} = \phi(x_i)$ for some $\phi(x)$. If the equation $x = \phi(x)$ has a root $\alpha$ and in the interval $J$ $\phi'(x)$ exists and satisfies $|\phi'(x)| \leq m < 1$ then for all $x_0 \in J$:

   (a) $x_n \in J$, $n = 0, 1, 2, ...,$

   (b) $\lim_{n\to\infty} x_n = \alpha,$

   (c) $\alpha$ is the only root in $J$ of $x = \phi(x)$.

   Prove a, b, and c given the initial assumptions.

7. $\mathbf{A} \in \mathbb{R}^{3\times3}$ has the eigenvalues $\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1$. Consider applying the power method with a shift in order to find $\lambda_1$. Show that the method will produce estimates that converge to $\lambda_1$ given appropriate conditions on the inital vector $\vec{v}^{(0)}$ and shift $\mu$. Explicitly state those conditions.

8. The cubic polynomial $p(x)$ interpolates the data $f(x_0)$, $f(x_1)$, $f'(x_1)$, and $f''(x_1)$ where $f \in C^4(-\infty, \infty)$. Let $x_0 < x_e < x_1$. Prove that:

$$\text{error} = f(x_e) - p(x_e) = \frac{f^{(iv)}(\xi)}{4!}(x_e - x_0)(x_e - x_1)^3, \qquad \xi \in [x_0, x_1].$$

Qualifying Exam for Numerical Analysis (Spring 2016)

Please show your work.

1) Suppose $r \in \Re$ is a simple zero of $f(x)$, where $f(x) \in C^2$ in a neighborhood of $r$. Discuss the conditioning of locating the root. Use the result to find the change in the largest zero of $f(x) = \prod_{i=1}^{n}(x-i)$ for a small perturbation in the coefficient of $x^n$.

2) The second derivative $f''(\bar{x})$ of a function may be approximated by using the values of the function at n equally spaced points $x_1, x_2, ..., x_n$, where $x_1 < x_2 < x_3 < ... < x_n$, $x_i - x_{i-1} = h, i = 2, 3, ..., n$, and $\bar{x} \in [x_1, x_n]$. That is, we may find $a_i, i = 1, 2, .., n$ such that

$$f''(\bar{x}) \approx a_1 f(x_1) + a_2 f(x_2) + ... + a_n f(x_n).$$

a) If a scheme with leading error term $O(h^2)$ is desired, how many points should be used?

b) Suppose $\bar{x}$ coincides with $x_2$, find the numerical scheme.

3) For the function $f(x) = \frac{1}{2-x}$ defined on [-1,1], use three interpolating polynomials of degree n to approximate it as described below. Find the error bounds for the three cases.

a) Use n+1 arbitrarily spaced distinct nodes on [-1,1] including the two endpoints.

b) Use n+1 equally spaced nodes on [-1,1] including the two endpoints.

c) Use n+1 Chebyshev nodes (i.e. zeros of Chebyshev polynomials of degree n+1) on [-1,1].

4) Consider the integral $\int_0^1 f(x)dx$, where $f(x)$ is continuous on [0,1] and $C^2$ on (0,1).

a) Construct a numerical method using a polynomial interpolating the integrand at two arbitrarily distinct points $x_0$ and $x_1$ on the interval [0,1]. That is, find $A_0$ and $A_1$ such that

$$\int_0^1 f(x)dx \approx A_0 f(x_0) + A_1 f(x_1).$$

Simplify the method when $x_0 = 0$ and $x_1 = 1$. What is the resulting method?

b) Derive an error bound of the above method for the general choice of $x_0$ and $x_1$.

c) Find the two nodes $x_0$ and $x_1$ such that the method becomes a Gaussian quadrature rule. Write down the corresponding values for $A_0$ and $A_1$ in this case.

d) Suppose the methods developed in a) and c) are used for $\int_0^1 p_n(x)dx$, where $p_n$ is a polynomial of degree n. For each of the methods, what is the maximum value of n such that the numerical method gives the exact result (ignoring rounding off error)?

5) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is a nonsingular n x n matrix. Let C be any n x n matrix and $\mathbf{c}$ be any vector in $\Re^n$. Let $\epsilon$ be a real number and $\mathbf{x}(\epsilon)$ be the solution of the perturbed system $(A + \epsilon C)\mathbf{x}(\epsilon) = \mathbf{b} + \epsilon\mathbf{c}$ in a neighborhood of $\epsilon = 0$. Let $\delta\mathbf{x} = \mathbf{x}(\epsilon) - \mathbf{x}$ where $\mathbf{x}$ is the solution to $A\mathbf{x} = \mathbf{b}$. Let $p \geq 1$ or $p = \infty$. Prove that

$$\frac{||\delta\mathbf{x}||_p}{||\mathbf{x}||_p} \leq \epsilon\kappa(A)\left(\frac{||C||_p}{||A||_p} + \frac{||\mathbf{c}||_p}{||\mathbf{b}||_p}\right) + o(\epsilon),$$

where $\kappa(A)$ is the condition number of the matrix A.

6) Let $f(x) \in C^3(-\infty, \infty)$, let $a$ and $b$ be two arbitrary points. Construct a sequence $(x_n), n = 1, 2, ...$, by the rule: $x_1 = a, x_2 = b$, and $x_{n+2}$ is the zero of the interpolating polynomial through the three points $(x_n, f(x_n)), ((x_{n+1} + x_n)/2, f((x_{n+1} + x_n)/2)), (x_{n+1}, f(x_{n+1}))$. Assume the sequence exists and converges to a real number $r \in (a, b)$. Let $e_n = |x_n - r|$.

a) Prove that $r$ is a zero of $f(x)$.

b) Show that $e_{n+2} \sim Ce_{n+1}^2$ with $C$ a positive constant, i.e., $e_{n+2}$ behaves as $Ce_{n+1}^2$ as n becomes sufficiently large.

Qualifying Exam for Numerical Analysis (Fall 2016)

Please show your work.

1) Let $R(n,0), n = 0,1,2,..,N$ (N is an integer greater than 3) denote a sequence of approximation to $\int_a^b f(x)dx$ ($f(x)$ is continuous) with $2^n$ subintervals of equal width $h = (b-a)/2^n$ by some numerical method. Suppose it is known that $\int_a^b f(x)dx = R(n,0) + c_2h^2 + c_3h^3 + c_4h^4 + ...$, where the coefficients $c_i, i = 2,3,4,...$ depend on $f(x)$ but not on $h$. Let $R(n,m), n = m, m+1, ..., N$ (m is an integer and $1 \le m \le N$) denote new sequences of more accurate approximations to the integral obtained from $R(n-1,m-1)$ and $R(n,m-1)$ by applying Richardson extrapolation to eliminate the leading error term in the approximations $R(k,m-1), k = m-1,...,N$. Find the equations for computing $R(n,1)$, $R(n,2)$, and $R(n,3)$.

2) Suppose the third derivative of a function $f(x) \in C^4(-\infty,\infty)$ at a point $x_0$, $f'''(x_0)$, is approximated in the following form:

$$f'''(x_0) \approx af(x_0-h) + bf(x_0) + cf(x_0+h) + df(x_0+2h),$$

where $h$ is a given real number and $a,b,c,d$ are constant coefficients to be determined. Find the coefficients $a,b,c,d$ such that the approximation is as accurate as possible. What is the order of accuracy of the approximation?

3) Given real numbers $x_i, a_i, b_i, c_i, i = 1,2,3$, where $x_i$ are distinct, find a polynomial $p(x)$ of degree at most 8 such that $p(x_i) = a_i, p'(x_i) = b_i, p''(x_i) = c_i$. (Do not try to solve a 9 by 9 linear system for the coefficients of the polynomial.)

4) Construct a 3-node Gaussian quadrature for the integral $\int_{-1}^0 f(x)dx$, where $f(x)$ is continuous on [-1,0], as follows:

$$\int_{-1}^0 f(x)dx \approx A_0f(x_0) + A_1f(x_1) + A_2f(x_2).$$

Find the three nodes $x_0, x_1, x_2$ and the corresponding weights $A_0, A_1, A_2$. Suppose $f(x)$ is a polynomial of degree n, what is the maximum value of n such that the Gaussian quadrature gives the exact result of the integral?

5) Consider the following linear system $A\mathbf{x} = \mathbf{b}$, with A and $\mathbf{b}$ given as below.

$$A = \begin{pmatrix} -1 & 10^6 \\ 2 & -3 \end{pmatrix}$$

and $\mathbf{b} = [10^6, 1]^t$.

a)Suppose $\mathbf{b}$ is perturbed by $\delta\mathbf{b}$ with the property that $||\delta\mathbf{b}||_\infty \leq 10^{-5}$. Find the upper bound for $\frac{||\delta\mathbf{x}||_\infty}{||\mathbf{x}||_\infty}$. Here $\mathbf{x}$ is the solution to the original linear system and $\delta\mathbf{x}$ is the perturbation to the solution due to the perturbation in $\mathbf{b}$.

b)Suppose A is perturbed by $\delta A$ with the property that $||\delta A||_\infty \leq 10^{-5}$. Find the upper bound for $\frac{||\delta\mathbf{x}||_\infty}{||\mathbf{x}+\delta\mathbf{x}||_\infty}$. Here $\mathbf{x}$ is the solution to the original linear system and $\delta\mathbf{x}$ is the perturbation to the solution due to the perturbation in A.

6) Let $f(x) \in C^4(-\infty, \infty)$ have a simple zero $r$. Construct an iterative numerical method to locate the root of the equation $f(x) = 0$, starting from an initial guess $x_0$:

$$x_{n+1} = \frac{1}{2}(a_{n+1} + b_{n+1}), \qquad n = 0, 1, 2, 3, ...$$

where

$$a_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

and

$$b_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)},$$

here the function $g(x) = \frac{f(x)}{f'(x)}$. Prove that if the sequence $\{x_n\}$ converges to $r$, then the convergence is cubic.

**Math 598 - Numerical Analysis**
**Qualifier Exam - January 2014**

**First and Last Name: ..............................................................**

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## You must provide SOME work for EACH of the sections below

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# 1 Linear Systems

1. Show that solving the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, with $\mathbf{A} \in \mathbb{R}^{n\times n}$ symmetric and positive definite and $\mathbf{b} \in \mathbb{R}^n$, is equivalent to minimizing the functional $\Phi(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T\mathbf{A}\mathbf{y} - \mathbf{y}^T\mathbf{b}$.

2. To solve the following block linear system

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B} \\ \mathbf{B} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{B}$ in $\mathbb{R}^{n\times n}$, and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{b}_1$ and $\mathbf{b}_2$ in $\mathbb{R}^n$, consider the following two methods:

(a) $\mathbf{A}_1\mathbf{x}^{(k+1)} + \mathbf{B}\mathbf{y}^{(k)} = \mathbf{b}_1, \quad \mathbf{B}\mathbf{x}^{(k)} + \mathbf{A}_2\mathbf{y}^{(k+1)} = \mathbf{b}_2$

(b) $\mathbf{A}_1\mathbf{x}^{(k+1)} + \mathbf{B}\mathbf{y}^{(k)} = \mathbf{b}_1, \quad \mathbf{B}\mathbf{x}^{(k+1)} + \mathbf{A}_2\mathbf{y}^{(k+1)} = \mathbf{b}_2$

Find sufficient conditions in order for the two schemes to be convergent for any choice of the initial data $\mathbf{x}^{(0)}$ and $\mathbf{y}^{(0)}$.

# 2 Root Finding

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function, with $f \in \mathcal{C}^\infty(\mathbb{R})$.

(a) define the method of fixed-point iterations to solve $f(x) = 0$

(b) study the well-posedeness of fixed-point iterations methods

(c) discuss the possible choices for stopping criteria

2. For the approximation of the zeros of the function

$$f(x) = \frac{2x^2 - 3x - 2}{x - 1}$$

consider the following fixed-point iterations methods:

(a) $x^{(k+1)} = g(x^{(k)})$, with $g(x) = (3x^2 - 4x - 2)/(x - 1)$

(b) $x^{(k+1)} = h(x^{(k)})$, with $h(x) = x - 2 + x/(x - 1)$

Analyze the convergence properties of the two methods and determine their order of convergence.

# 3 Polynomial Interpolation

1. Prove the following statement:

   *Let $x_0, x_1, ..., x_n$ be n+1 distinct nodes and let $x$ be a point belonging to the domain of a given function $f : \mathbb{R} \to \mathbb{R}$. Let $\Pi_n f$ be the polynomial of degree $n$ satisfying $\Pi_n f(x_i) = f(x_i)$, for $i = 0, ..., n$. If $f \in \mathcal{C}^{n+1}(\mathbb{R})$, then the interpolation error at the point $x$ is given by*

   $$E_n(x) = f(x) - \Pi_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} w_{n+1}(x) \tag{1}$$

   *where $\xi \in \mathbb{R}$ and $w_{n+1}$ is the nodal polynomial of degree $n+1$.*

2. Let $f(x) = \sin x$, and let $x_0 = 0$, $x_1 = \pi/6$, $x_2 = \pi/2$.

   (a) define the nodal polynomial $w_3$

   (b) determine the interpolating polynomial $\Pi_2 f$

   (c) calculate the interpolation error at $x = \pi/4$

   (d) interpret the interpolation error using the theoretical estimate (1)

# 4 Integration

1. Define the trapezoidal quadrature formula and derive its quadrature error.

2. Let $f \in \mathcal{C}(\mathbb{R})$ and let

   $$I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$$

   be a Lagrange quadrature formula on $n+1$ nodes.
   Compute the degree of exactness $r$ of the formulae:

   (a) $I_2(f) = \frac{2}{3}\left[2\, f\left(-\frac{1}{2}\right) - f(0) + 2\, f\left(\frac{1}{2}\right)\right]$

   (b) $I_4(f) = \frac{1}{4}\left[f(-1) + 3\, f\left(-\frac{1}{3}\right) + 3\, f\left(\frac{1}{3}\right) + f(1)\right]$

**Math 598 - Numerical Analysis**
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**You must provide some work**
**for EACH of the sections below**

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# 1 Norms and Inner products

1. Consider the subtraction $x = a - b$ of two real numbers $a$ and $b$ such that $a \neq b$. Suppose that $\hat{a}$ and $\hat{b}$ are the result of making a *relative* perturbation $\Delta a$ and $\Delta b$ to $a$ and $b$. Find the relative error of $\hat{x} = \hat{a} - \hat{b}$ as an approximation to $x$ and hence find the condition number for the operation of subtraction with respect to $\Delta a$ and $\Delta b$. *Assume that all calculations are done in exact arithmetic.*

2. Let $\mathbf{D} \in \mathbb{R}^{n \times n}$ with $\mathbf{D} = diag(d_1, d_2, ..., d_n)$. Prove that the matrix p-norm is such that
$$\|\mathbf{D}\|_p = \max_{1 \leq i \leq n} |d_i|$$
for all $1 \leq p \leq \infty$.

# 2 Linear Systems

Consider the following iterative method to solve the linear system $\mathbf{Ax} = \mathbf{b}$:

*Given $x_0 \in \mathbb{R}^n$, compute $\mathbf{x}^{(k+1)}$ as*

$$\mathbf{B}(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) = \omega(\mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}) \qquad \forall k > 1$$

*where $\omega$ is a positive constant and $\mathbf{A}$ and $\mathbf{B}$ are symmetric, positive definite $n \times n$ matrices such that*

$$\mathbf{B}^{-1}\mathbf{A} = \mathbf{I} + \mathbf{C}, \quad \textit{with } \|\mathbf{C}\|_2 = \gamma.$$

1. Is the method consistent? If yes, is the method consistent for any choice of $\omega$ and $\mathbf{B}$? If not, provide a range.

2. Is the method convergent for any choice of $\omega$ and $\mathbf{B}$? If not, provide a range.

3. Derive a relationship between the error vector $\mathbf{e}^{(k)} = \mathbf{x} - \mathbf{x}^{(k)}$ and the difference between iterates $\mathbf{d}^{(k)} = \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$. Under which conditions is the stopping criterium $\|d^{(k)}\|_2 < \epsilon$ a good choice?

# 3 Root Finding

1. To find the root of $f(x) = 0$, let $g(x) = x + cf(x)$ for some constant $c \neq 0$. Let $\alpha$ be a root of $f(x)$ and $f'(\alpha) = 2$. Consider the fixed point iterations:
$$x^{(k+1)} = g(x^{(k)}).$$
and assume that $x^{(0)}$ is close enough to $\alpha$ and that $f$ is smooth.
   (a) For what values of $c$ will the sequence converge to $\alpha$?
   (b) For what values of $c$ will the convergence be quadratic?

2. Describe the chord's method and the Newton's method as fixed point methods and determine their order of convergence using Ostrowski's theorem.

# 4 Polynomial Interpolation

Let $x_o, x_1, ..., x_n$ be $n + 1$ distinct nodes and let $x$ be a point belonging to the domain of the function $f \in \mathcal{C}^{n+1}(I_x)$, where $I_x$ is the smallest interval containing $x_o, x_1, ..., x_n$ and $x$. Let $\pi_n f$ be the Lagrange polynomial interpolating $f$ at $x_i$, for $i = 0, ..., n$.

Show that

1. $\pi_n f$ is unique;

2. the interpolation error $E_n(x)$ is given by
$$E_n(x) = f(x) - \pi_n f(x) = \frac{f^{n+1}(\xi)}{(n+1)!}\omega_{n+1}(x),$$
for some $\xi \in I_x$, where $\omega_{n+1}(x)$ is the nodal polynomial of order $n + 1$.

# 5 Integration

1. Given a scalar function $f$, derive the midpoint and trapezoidal rules to approximate $\int_a^b f(x)dx$ by replacing $f(x)$ with its appropriate polynomial interpolant.

2. Show that the integration error for the trapezoidal rule is approximately two times larger than the integration error for the midpoint rule.

# Math 598 - Numerical Analysis
# Qualifier Exam - August 2013

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# 1 Norms and Inner products

Consider the mathematical problem:

$$\textit{Find } x \in \mathbb{R} \textit{ such that } F(x,d) = 0 \;\; \forall d \in D \subset \mathbb{R}, \tag{1}$$

where $F: \mathbb{R} \times D \to \mathbb{R}$ is a $C^\infty$ mapping.
Consider the following numerical method to find an approximate solution of the mathematical problem (1):

$$\begin{gathered} \textit{Given } x_0, \textit{for } n \geq 1 \textit{ find } x_n \in \mathbb{R} \textit{ such that} \\ F_n(x_n, x_{n-1}, d) = 0 \;\; \forall d \in D. \end{gathered} \tag{2}$$

1. Define when the numerical method (2) is
   (a) consistent;
   (b) strongly consistent;
   (c) well-posed (or stable);
   (d) convergent.
2. What is the relationship between consistency, stability and convergence?

# 2 Linear Systems

Consider the matrix $\mathbf{A} \in \mathbb{R}^2$ with $a_{11} = a_{22} = 1$, $a_{12} = \gamma$, and $a_{21} = 0$.

1. Show that $K_\infty(\mathbf{A}) = K_1(\mathbf{A}) = (1+\gamma)^2$.
2. Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b}$ is such that $\mathbf{x} = (1-\gamma, 1)^T$ is the solution of the system.
   (a) Find a bound for $\|\delta\mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty$ in terms of $\|\delta\mathbf{b}\|_\infty / \|\mathbf{b}\|_\infty$ assuming that $\delta\mathbf{b} = (\delta_1, \delta_1)^T$.
   (b) Is the problem well- or ill- posed?

# 3 Root Finding

Consider the following fixed-point method (known as Steffensen's method) to approximate the zeros of the equation $f(x) = 0$, with $f \in C^\infty(\mathbb{R})$.

*Given $x_0$, for $k \geq 0$ compute*

$$x_{k+1} = x_k - \frac{f(x_k)}{\phi(x_k)}, \quad \textit{with} \quad \phi(x_k) = \frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}. \tag{3}$$

Determine the order of convergence of such method.

# 4 Polynomial Interpolation

Let $T_h$ be a partition of the interval $(a, b) \in \mathbb{R}$ and let $x_j$, $j = 0, ..., k$ be the equidistant points in this partition. Thus $x_0 = a$, $x_k = b$, and $x_{j+1} - x_j = h$ for $j = 0, .., n-1$. Consider the function $f \in C^2(a, b)$ and its polynomial interpolation $\Pi_h^1 f$ of degree 1.

Show that

$$\|f - \Pi_h^1 f\|_{L^2(a,b)} \leq h^2 \|f''\|_{L^2(a,b)}, \tag{4}$$

where $f''$ is the second derivative of $f$.

# 5 Integration

Consider the definite integral $I(f) = \int_a^b f(x)dx$, with $f \in C^\infty(a, b)$.

1. Provide definitions of:
   (a) the Lagrange quadrature formula to obtain the approximation $I_n(f)$ of the integral $I(f)$;
   (b) the degree of exactness of a Lagrange quadrature formula;
   (c) the order of infinitesimal of a Lagrange quadrature formula.
2. Specify the quadrature formula, the order of exactness and the order of infinitesimal for the two cases when $n = 0$ and $n = 1$.