Differential Geometry qualifying exam 562 — January 2019

Show all your work for full credit

1. (a) Show that the set $M \subset \mathbb{R}^3$ defined by the equation $(1-z^2)(x^2+y^2) = 1$ is a smooth submanifold of $\mathbb{R}^3$.

(b) Compute a vector field in $\mathbb{R}^3$ that is normal to M.

(c) Define a vector field on $\mathbb{R}^3$ by $V = z^2x\frac{\partial}{\partial x} + z^2y\frac{\partial}{\partial y} + z(1-z^2)\frac{\partial}{\partial z}$. Show that the restriction of V to M is a tangent vector field to M.

2. Let M be an n-dimensional manifold and let ω be a differential form on M of even degree. Show that the form $\omega \wedge \mathrm{d}\omega$ is always exact.

3. (a) Consider a smooth manifold M. Recall that if X is a vector field and f, g are smooth functions, then Xg represents the derivation X applied to the function g and subsequently fXg represents the product between the function f and the function Xg.
Show that the Lie bracket $[\cdot\,,\cdot]$ satisfies

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

(b) Compute the Lie bracket of the vector fields $X = (2x+y)\frac{\partial}{\partial x} + (x^2-y)\frac{\partial}{\partial y}$, and $Y = xy^3\frac{\partial}{\partial x} + x^3y^2\frac{\partial}{\partial y}$.

4. Let ω denote the standard area form on $S^2 \subset \mathbb{R}^3$ (with the orientation determined by the outward-pointing normal vector). Here you can think that the form is obtained by contracting $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ with the outward normal vector, where x, y, z denote the standard rectangular coordinates for $\mathbb{R}^3$.

Show that for a non-negative integer n, the form $\alpha = z^n\omega$ is exact, if and only if n is odd. You may use that in the standard spherical coordinates $(x, y, z) \mapsto (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, the form ω equals $\sin\phi\, \mathrm{d}\phi \wedge \mathrm{d}\theta$.

5. Show that a smooth submersion from a compact smooth manifold to a connected smooth manifold must be surjective.

6. Let G be a Lie group of dimension n. A vector field X on G is called left-invariant if $X(gh) = \mathrm{d}_hL_g(X(h))$ for all $g, h \in G$. (Here, $L_g : G \longrightarrow G$ is left multiplication by $g \in G$.) Show that the space of left-invariant vector fields is a real vector space of dimension n, invariant under the Lie bracket.

QUALIFYING EXAM

Janury 2017

MATH 562 - Prof. Buse

(1) Define the complex projective space $\mathbb{CP}^n$ as the quotient space of $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ by the equivalence relation $(z_0, \ldots, z_n) \equiv (\lambda z_0, \ldots, \lambda z_n)$ for any complex number $\lambda \neq 0$. Denote such an equivalence class $[z_0 : \ldots : z_n]$. There is a quotient map $\pi : \mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\} \longrightarrow \mathbb{CP}^n$ taking $(z_0, \ldots, z_n)$ to $[z_0 : \ldots : z_n]$. One endows $\mathbb{CP}^n$ with the quotient topology by declaring that a set U is open in $\mathbb{CP}^n$ if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$.

Show that $\mathbb{C}P^n$ is a smooth manifold by using the definition. (You do not need to show that the topology is well defined.)

(2) Show that $\mathbb{CP}^1$ is diffeomorphic to the sphere S^2.

(3) Consider the vector fields on $\mathbb{R}^2$ given by $E = x\frac{\partial}{\partial y}$, $F = y\frac{\partial}{\partial x}$ and $H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

(a) Prove that the span of E, H, and F is closed under the commutator(Lie bracket) .

(b) Consider a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x, y) = x^j y^k$, for j and k positive integers. With the usual understanding of a vector field as a derivation, compute $E(f)$, $F(f)$, and $H(f)$.

(4) (a) Given two smooth orientable manifolds M^m and N^n show that the product $M \times N$ is orientable.

(b) Is the converse true? Prove your answer.

(5) For each of the following statements determine with proof if it is true or false:

(a) Any smooth 1-form on S^1 can be extended to a smooth 1-form in $\mathbb{R}^2$.

(b) Any smooth, closed 1-form on S^1 can be extended to a smooth, closed 1-form in $\mathbb{R}^2$.

(6) Let M^{n+1} be a compact, orientable, smooth $(n+1)$-manifold with boundary X. The orientation of X as an n-dimensional manifold is induced from the orientation of M. Show, using Stokes' theorem, that there is no smooth retraction from M to its boundary X. Recall that a retraction is a map $F : M \longrightarrow X$ such that $F_{|X} = \mathrm{id}_X$.

QUALIFYING EXAM
AUGUST 2016
MATH 562 - Prof. Buse

1. Let M, N be smooth manifolds without boundary.
a) Define a *smooth submersion* between M and N.
b) Show that a submersion is an open map.
c) Show that if M is compact and N is connected then any $f : M \longrightarrow N$, a smooth submersion must be surjective.
d) Is the statement in c) still true if M is not compact?

2. a) Define the concept of orientability on a smooth manifold.
b) Show that the tangent bundle TM of any smooth manifold M without boundary is always an orientable manifold (you only need to show the orientability, not the manifold structure).

3. Let P be a compact smooth oriented n-dimensional manifold without boundary. Show that for any $(n-1)$ smooth form ω on P there exists a point $p \in P$ such that $\mathrm{d}\omega(p) = 0$.

4. Define the following 2-form on $N := \mathbb{R}^3 \setminus \{0\}$:

$$\alpha = \frac{x\mathrm{d}y \wedge \mathrm{d}z - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{3/2}}$$

a) Show that $\mathrm{d}\alpha = 0$.
b) Show that $\int_{x^2+y^2+z^2=r^2} \alpha$ is independent of the constant r.

5. Let X and Y be manifolds, and let U, Z be submanifolds of Y.
a) Assume that $f : X \longrightarrow Y$ is a smooth map, transversal to Z in Y, so that $W := f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $\mathrm{d}f : T_x(X) \longrightarrow T_{f(x)}(Y)$.
b) Assume that U is transversal to Z. Show that for $y \in U \cap Z$ it holds that $T_y(U \cap Z) = T_y(U) \cap T_y(Z)$.
c) Is the statement in b) still true without the transversality condition?

6. Show that the group $O(n)$ of orthogonal n-dimensional square matrices is a smooth manifold, and compute its dimension. Recall that a matrix Q is orthogonal if $Q^T Q = I$.

IUPUI Qualifying Exam
Math 56200: Differential Geometry and Topology

January 2015 **Daniel Ramras**

(1) Let M be a smooth, compact n–dimensional manifold without boundary ($n > 0$) and let N be a smooth manifold (also without boundary). Assume that there exists a submersion $p\colon N \to \mathbb{R}$.

Prove that for each smooth function $f\colon M \to N$, there exist at least two distinct points $x, y \in M$ such that $T_x f\colon T_x M \to T_{f(x)}N$ and $T_y f\colon T_x M \to T_{f(y)}N$ are *not* surjective.

(2) Let M be a smooth manifold without boundary. Let X be a C^∞ vector field on M, and let $f, g\colon M \to \mathbb{R}$ be C^∞ functions.

a) State the definition of the Lie derivative $\mathcal{L}_X(g)$.

b) Prove, directly from your definition in part a), that if $\mathcal{L}_X(g) = 0$ then $\mathcal{L}_{fX}(g) = 0$ as well.

(3) (a) Prove that $H_n = \{(v, w) \in \mathbb{R}^n \times \mathbb{R}^n : \langle v, w\rangle = 1\}$ is a smooth manifold. Here $\langle v, w\rangle = v \cdot w$ is the standard inner product on $\mathbb{R}^n$.

(b) Is H transverse to the diagonal $\Delta = \{(v, v) \in \mathbb{R}^n \times \mathbb{R}^n\}$? Prove your answer.

(4) Consider the vector field $\frac{\partial}{\partial x_1}$ on $\mathbb{R}^2$. Let $\psi\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection map

$$\psi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

(where $N = (0, 0, 1)$).

(a) Show that the vector field $V = \psi^*(\frac{\partial}{\partial x_1})$ extends to a smooth vector field X on the entire sphere S^2, with the property that $X_p = 0$ if and only if $p = N$.

(b) Let $\gamma\colon (-a, a) \to S^2$ be an integral curve of V. Show that γ extends to an integral curve defined on all of $\mathbb{R}$, and prove that

$$\lim_{t\to\infty} \gamma(t) = \lim_{t\to-\infty} \gamma(t) = N.$$

(5) Let X, Y, and Z be compact oriented k–dimensional manifolds without boundary, and consider smooth maps $f\colon X \to Y$ and $g\colon Y \to Z$.

Prove that

$$\deg(g \circ f) = \deg(g)\deg(f).$$

(6) Let A be a 2×2 matrix with real entries, and consider the 1–form ω_A on S^1 defined by $\omega_x(v) = \langle v, Ax\rangle$, where $v \in T_x(S^1) \subset \mathbb{R}^2$.

(a) Give a formula for $\displaystyle\int_{S^1} \omega$ in terms of the entries of A.

(b) Characterize those matrices A for which the form ω_A is closed but not exact.

IUPUI Qualifying Exam
Math 56200: Differential Geometry and Topology

August 2014 **Daniel Ramras**

(1) Let A and B be $n \times n$ matrices with real entries, and consider the vector fields α and β on $\mathbb{R}^n$ defined by $\alpha_{\mathbf{x}} = A\mathbf{x}$ and $\beta_{\mathbf{x}} = B\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$. Show that the Lie bracket $[\alpha, \beta]$ also has the form $[\alpha, \beta]_{\mathbf{x}} = C_{\mathbf{x}}$ for some $n \times n$ matrix C, and compute C in terms of A and B.

(2) Let M and N be smooth manifolds. Prove that $T(M \times N)$ is diffeomorphic to $TM \times TN$, where $T(\cdot)$ denotes the tangent bundle. Make sure to explain why the maps used in your solution are smooth.

(3) Consider the smooth map $\det\colon M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ given by sending each real-valued $n \times n$ matrix A to its determinant $\det(A)$. The derivative of det at the identity matrix $I \in M_{n\times n}(\mathbb{R})$ is a linear transformation $D_I(\det)\colon M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ (using the standard identifications of the vector spaces $M_{n\times n}(\mathbb{R})$ and $\mathbb{R}$ with their tangent spaces).
   (a) Prove that $D_I(\det)(A) = \operatorname{trace}(A)$. (Hint: consider the cofactor expansion formula for the determinant.)
   (b) Consider the special linear group $\mathrm{SL}_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det(A) = 1\}$. Prove that $\mathrm{SL}_n(\mathbb{R})$ is a smooth manifold, and calculate its dimension.
   (c) Calculate $T_I\mathrm{SL}_n(\mathbb{R})$ as a subspace of $M_{n\times n}(\mathbb{R})$.

(4) Let Y and W be manifolds and let X and Z be submanifolds of Y that intersect transversally. Prove that if $f\colon W \to Y$ is a submersion, then $f^{-1}(X)$ and $f^{-1}(Z)$ intersect transversally in W.

(5) Compute the integral

$$\int_{S^2} 2xyz\,dx \wedge dy + (yz + xy^2)\,dx \wedge dz + xz\,dy \wedge dz.$$

You may assume that S^2 is oriented as the boundary of the unit ball $D^3 \subset \mathbb{R}^3$.

(6) Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$.
   (a) Consider the smooth map $\tau\colon S^1 \times S^1 \to S^1 \times S^1$ defined by $\tau((x,y),(z,w)) = ((x,-y),(z,-w))$. Is this map orientation preserving or orientation reversing? Justify your answer.
   (b) Prove that the map $f\colon S^1 \times S^1 \to S^2$ defined by

$$f((x,y),(z,w)) = \frac{1}{\sqrt{x^2 + z^2 + (yw)^2}}(x, z, yw)$$

is *not* homotopic to a constant function.

IUPUI Qualifying Exam, Math 562,
Introduction to Differential Geometry and Topology

August 2012 **Roland Roeder**

You must provide detailed reasoning to support your claims. You may use any theorems (but not examples nor exercises) in the textbook to support your arguments. However, if a problem is itself part of a theorem in the textbook, you must provide proofs.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = (x^2 + y^2 - 4)^2 + z^2 - 1$ and let $M = f^{-1}(0)$.
   (a) Prove that M is a manifold.
   (b) Prove that M is diffeomorphic to the two dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.
   (c) For what values of $r > 0$ does the cylinder $x^2 + y^2 = r^2$ intersect M transversally?

2. Let M be a compact manifold without boundary. Show that there is no submersion $f : M \to \mathbb{R}$.

3. Let $M, N \subset \mathbb{R}^{k+1}$ be disjoint compact manifolds without boundary satisfying $\dim(M) + \dim(N) = k$. Let

$$\lambda : M \times N \to \mathbb{S}^k \text{ be given by } \lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{x} - \boldsymbol{y}}{\|\boldsymbol{x} - \boldsymbol{y}\|}.$$

   The *linking number* between M and N is defined by $lk(M, N) = \deg(\lambda)$.
   (a) Prove that if M is the boundary of an orientable manifold X that is disjoint from N, then $lk(M, N) = 0$.
   (b) Let $M = \{x^2 + y^2 = 1, z = 0\}$ and $N = \{x = 0, (y - 1)^2 + z^2 = 1\}$, considered as submanifolds of $\mathbb{R}^3$. Compute $lk(M, N)$.

4. Suppose that $f : \mathbb{R}^k \to \mathbb{R}^k$ has a fixed point at $\boldsymbol{x}_0$ and let B be a closed ball centered at $\boldsymbol{x}_0$ containing no other fixed point of f. Let f_1 be some smooth map equal to f outside some compact subset of $\text{Int}(B)$, and having only Lefschetz fixed points in B. Prove that

$$L_{\boldsymbol{x}_0}(f) = \sum_{\boldsymbol{x} \in B, f_1(\boldsymbol{x}) = \boldsymbol{x}} L_{\boldsymbol{x}}(f_1).$$

5. Let U be a compact region in $\mathbb{R}^3$ with smooth boundary. Show that the volume of U is given by

$$\int_{\partial U} \frac{1}{3}(z dx \wedge dy + y dz \wedge dx + x dy \wedge dz)$$

6. Show that the form

$$\omega = \frac{x dy - y dx}{x^2 + y^2}$$

   on $\mathbb{R}^2 \setminus \{(0, 0)\}$ is closed but not exact.